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Correspondences Determined by the Bitangents of a Quartic.

By J. R. Conner.

Part I.

Introduction.

The connection of the plane quartic curve with the figure of the general net of quadrics in space was first pointed out by Hesse*; he considered the net of quadrics in the form

$$(a\lambda)(\alpha x)^2=0\,(3\ \lambda\text{'s},\ 4\ x\text{'s})\,; \tag{1}$$

writing the discriminant of (1) considered as a form in x,

$$(a\lambda)(b\lambda)(c\lambda)(d\lambda)\,|\alpha\beta\gamma\delta|^2=0, \tag{2}$$

and regarding the λ's as coordinates of a point in a plane π, (2) is the equation of a quartic curve in π, and the points of this curve are in (1, 1) correspondence with nodal quadrics of the system (1). In other words, (1) defines a univocal correspondence between points of π and quadrics of the net; corresponding to the quadrics of a pencil chosen from the net is a line of π, and to the four nodal quadrics of the pencil the points of intersection of the line with the quartic (2). Hesse showed that if (1) is a general net, (2) is a general quartic. To a double tangent of (2) corresponds a pencil of quadrics (1) which has the property that the nodal quadrics coincide in pairs; such a pencil is that containing one line of the twenty-eight joining in pairs the base-points of the net (1). If a_i are these base-points, the base-curve of the pencil containing a_ia_j is a_ia_j taken with the cubic curve on the other six a's; the two quadric cones corresponding to the points of contact of the associated double tangent are the cones projecting this cubic from its two points of intersection with a_ia_j. The double tangents of (2) are thus associated in a one-one way with the twenty-eight lines joining the eight base-points of (1), and this is the basis for the duad notation for the double tangents of a quartic, each being represented by two symbols chosen from eight.

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* Hesse, *Journal für Mathematik,* Vol. XLIX (1855).

In what follows we prefer to regard the equation (1) as determined in the following manner*: isolating one of the base-points of the net, say a_0, there is determined a one-to-one relation between the quadrics of the net and the bundle of planes on a_0; thus, a given quadric has a definite tangent plane at a_0, and a given plane on a_0 has a definite tangent quadric at a_0; this being true without exception, there is a collinear relation between the system of quadrics and the system of planes. The planes of the bundle on a_0, or, better, their lines of intersection with an arbitrary plane ω, take the place of the plane π in Hesse's discussion. It is our object, by means of the space figure, to prove the existence of certain Cremona transformations and other noteworthy correspondences associated with sets of the nodes of the quartic of lines thus obtained. The intimate connection between the plane and space figures makes many geometrical relations in both intuitive which might otherwise be difficult to prove. From our point of view there is another special convenience in determining the quartic in this way as a quartic of lines, for the correspondences arising appear as point to point correspondences. The method is almost exclusively geometrical, though the geometrical notions lead to analytical results whose possibilities we use to some extent, but make no attempt to exhaust.

In the first part of the paper attention is given to the correspondences themselves as they arise from the figure in space, and independently of their connection with the nodes of the quartic; then by means of these correspondences certain of their singular points are identified with the nodes of the quartic determined in the manner above specified. The correspondences are closely associated with configurations arising from the grouping of the nodes, and this feature is considered. Especial attention may be called to the Cremona involution of order 15 in space discussed in § 1 and to its use in the sequel. It is the analogue of the Bertini 17-ic transformation† in the plane, and is interesting for its own sake. We take occasion also to show how there arises from the figure in space a Kummer surface which has the property that its enveloping cone from a_0 is exactly our quartic of lines.

§ 1. *The 15-ic Cremona Transformation in Space.*

Let seven points, $a_0, a_1, \ldots, a_6$, be given in general position in space. They determine a 15-ic Cremona transformation which, as has been remarked in the Introduction, is the extension to space of the involutory 17-ic transformation in the plane. In order that the analogy may be seen more clearly we summarize the latter briefly. The 17-ic transformation is determined when its eight singular points α_i are given. There is a pencil of cubic curves "c" on the points α_i and ∞^3 sextics "s" with nodes at these points. There is a unique sextic s with triple point at one of the α's. Sextics s meet a definite cubic c in points of a g_2^1 on this cubic; thus, if x is a general point of the plane, sextics s on x pass through a definite point y, the partner of x in the g_2^1 on the unique cubic c on x; x and y may easily be seen to be partners in an involutory 17-ic Cremona transformation which has α_i as six-fold singular point, the corresponding fundamental curve being the unique sextic s with triple point at α_i. The Jacobian, j, of one sextic s and two cubics c is the locus of further possible nodes of sextics s; any node of a proper sextic s not at α_i must lie on j. The curve j is of order 9 and has triple points at α_i with the same tangents at α_i as the sextic s with triple point there. It is a locus of fixed points of the transformation. The transformation has besides an isolated fixed point, the ninth base-point of cubics c.

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* Compare Frobennis, *Journal für Mathematik,* Vol. XCIX, p. 275.

† Sturm, "Die Lehre von den geometrischen Verwandtschaften," Vol. IV, p. 105.

In space, we have a net of quadrics on the seven points a; we call these "quadrics Q" and indicate the eighth base-point by a. The general quartic surface in space has thirty-four constants, and since it is four linear conditions on a surface in space to have a node at a given point, there is a linear system of $\infty^{34-7\cdot 4}$ or ∞^6 quartic surfaces with nodes at a_i; we call these "quartics C." It is ten linear conditions on a surface to have a triple point at a given point, and hence there is a unique quartic C with triple point at a_i; we call this surface C_i. There are seven such surfaces. The ∞^2 elliptic quartic curves on a_i (and a) we call "curves E;" these are base curves of pencils of quadrics Q, and every quadric Q may be regarded as a locus of curves E,—it contains ∞^1 such curves. The degenerate curves E are of special importance for us. A line joining two points a taken with the norm curve on the other six a's (this curve is bisecant to the line) is a degenerate curve E. There are twenty-eight of these degenerate curves. The norm curves on a_0 taken with five of the other a's are conveniently separated into a set of six and a set of fifteen. We indicate the curve on $a_0, a_2, \ldots, a_6$ by r_1 (there are six such curves) and the curve on $a_0, a_3, \ldots, a_6$ by r_{12}, there are fifteen of these. We add r_0, the norm curve on $a_1, a_2, \ldots, a_6$.

A curve E is met by quartics C in sixteen points of which fourteen fall at a_i; the two variable points are pairs of a g_2^1 on E, either point of such a pair determining the other uniquely. Since there is a unique quartic E on the general point, x, of space, it is seen that *all quartics C on x pass through a second* *point y, x and y lying on the same curve E and forming a pair of a definite g_2^1 on this curve.* The points x and y are partners in the Cremona transformation in space which we wish to consider. This transformation we call T.

From the way in which T is defined follows the invariance of curves E under T; since quadrics Q are loci of curves E they are invariant, and the invariance of quartics C follows also from the definition. If

$$C=0,\ Q_i=0$$

are the equations of a general quartic C and three linearly independent quadrics Q, then any surface of the form

$$(\alpha Q)^{n-2}C+(\beta Q)^n=0 \tag{1}$$

is invariant under T. The surface (1) is of order $2n$ with n-fold points at a_i, though not the most general such surface; it is impossible to express the surface J to be mentioned presently in this form; J is, however, invariant under T, and every sextic surface with triple point at a_i may be built from J and the form (1) for $n=3$. Thus every sextic surface with triple point at a_i is invariant under T.

It is unnecessary for us to pursue further the general question of surfaces invariant under T; certain identities exist among powers of forms like J, Q and C which make the general treatment of the subject rather long. We shall have special use later for those invariant surfaces of the form $(\beta Q)^n=0$.

Recalling the definition of T by means of quartics C, observe now that quartics C on a general point of a line $a_i\,a_j$ (we call this p_{ij}) contain p_{ij} entirely; that quartics C on a general point of a curve r_i contain r_i entirely. Also, bearing in mind the relation of curves E to T, note that curves E meet a surface C_i in only one variable point, the other intersection falling always at a_i.

Indicating for the moment a point of a surface of curve S by $[x, S]$, and keeping in mind the remarks of the above paragraph, we have the following relations which characterize T:

$$T\,a_i=C_i, \tag{1}$$
$$T\,[x, p_{ij}]=p_{ij}, \tag{2}$$
$$T\,[x, r_i]\ =r_i, \tag{3}$$
$$T\,E=E, \tag{4}$$
$$T\,Q=Q, \tag{5}$$
$$T\,C=C. \tag{6}$$

Equations (1), (2), (3) define all possible singular points of T. The points a_i are quartic singular points, the corresponding fundamental surfaces being C_i;

points of p_{ij} and r_{ij} are singular points such that to a point of such a curve corresponds the whole curve.*

Equations (4), (5), (6) tell the rest of the story. From (1) we have that the transform of a general line passes four times through a_i, and that from the transform of a surface passing through a_i, C_i factors. If n is the order of the transformation, we have from (5)

$$2n - 7 \cdot 4 = 2,$$

whence $n = 15$. If m is the order of multiplicity at a_i of the transform of a plane, we have from (4)

$$15 \cdot 4 - 7m = 4,$$

whence $m = 8$. Hence,

a) *The transformation T is an involutory Cremona transformation of order* 15. *The points a_i are singular points with C_i as corresponding fundamental surfaces; points of p_{ij} are singular points with p_{ij} as corresponding lines; points of r_i are singular points with r_i as corresponding curve. The transform of a general plane passes* 8*-fold through a_i, contains r_i triply and p_{ij} singly. If a plane passes through a_i, C_i factors from its transform. Analogous theorems are true for the transforms of surfaces of any order. The transform of a curve of order n is in general of order* $15n$ *with* $4n$*-fold points at a_i. The order of the transform is reduced by one, three or eight, respectively, if the curve meets p_{ij} or r_{ij}, or passes through a_i.*†

Consider two corresponding points of T, x, y say. The curve E through x and y carries a pencil of quadrics Q, one of which contains the line xy entirely. It follows that

b) *Lines joining pairs of corresponding points of T are in the cubic complex of generators of quadrics Q.*

The conditions that two points ζ and η be apolar to all quadrics Q determine the well-known cubic Cremona involution in space, Z say, and the lines joining corresponding points of Z are generators of quadrics Q. If l is a line of the complex of generators, curves E bisecant to l (of course these curves lie on the quadric Q containing l) mark on l an involution of which the fixed points, points of contact of curves E with l, are the points ζ, η on l. Hence,

c) *The points xy, $\xi\eta$ on a line of the complex of generators of quadrics Q are harmonic pairs, where $Tx = y$, $Z\xi = \eta$.*

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* Sturm, "Die Lehre von den geometrischen Verwandtschaften," Vol. IV, p. 415.

† It seems unnecessary to insert here the usual arguments by which this conclusion is reached from the preceding; compare the volume of Sturm already referred to.

If a quartic C have a node at point x, the correspondent, y, of x by T falls at x. The point x is then either at a or on the Cayley* dianodal surface,

$$J \equiv \begin{vmatrix} \frac{\partial C}{\partial x_1}, & \frac{\partial Q_1}{\partial x_1}, & \frac{\partial Q_2}{\partial x_1}, & \frac{\partial Q_3}{\partial x_1} \\ \frac{\partial C}{\partial x_2}, & \frac{\partial Q_1}{\partial x_2}, & \cdots & \cdots \\ \frac{\partial C}{\partial x_3}, & \frac{\partial Q_1}{\partial x_3}, & \cdots & \cdots \\ \frac{\partial C}{\partial x_4}, & \frac{\partial Q_1}{\partial x_4}, & \cdots & \cdots \end{vmatrix} = 0,$$

where $C=0$ is a general quartic C and $Q_i=0$ are three linearly independent quadrics Q. It follows from the theory of Jacobians that J is a sextic surface with triple points at a_i and having at a_i the same tangent cone as C_i. These facts which are important in what follows, we state in the theorem

d) *The sextic dianodal surface J is a locus of fixed points of T and a is an isolated fixed point. The surface J has triple points at a_i with the same tangent cubic cones as C_i at these points.*

The plane π_{123} on a_1, a_2, a_3 is transformed into a surface of order $15-3\cdot 4=3$; this is the cubic surface with nodes at a_0, a_4, a_5, a_6 and passing through a_1, a_2 and a_3. The surfaces C_1, C_2 and C_3 which factor out of the transform of π_{123} make up the necessary multiplicity at a_i eight of the transform of a plane by T. The cubic surface meets π_{123} in a cubic curve which is invariant under T. The plane π_{123} and its transform together make up a quartic C which may be taken as the C in the expression for J.†

A set of lines invariant under T are the lines $a_i\,a$. For such a line meets C_i at one and only one point not at a_i, and this means that its transform passed once through a_i. The line is bisecant to r_i. On it a is fixed under T. Its transform is of order $15-8-2\cdot 3=1$ (Th. a)); hence, it is invariant under T. Corresponding points of T on a_ia are pairs of an involution, the points l_i and m_i in which r_i meets a_ia forming such a pair. The double points of the involution are a and its harmonic conjugate as to l_i and m_i, a point in general different from a_i since a_ia, l_im_i are not in general harmonic pairs. We state this in another way more suitable for our purpose. If the second double point on a_ia is at a_i, a_ia must touch J at a_i, and conversely. Therefore,

e) *The lines a_ia are invariant under T. They do not touch J at a_i.*

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* Cayley, *Proceedings of the London Mathematical Society,* Vol. III, p. 19.

† Pascal, "Repertorium der höheren Mathematik," Vol. II (1902), p. 297.

The following may be of interest in passing: The Geiser transformation in the plane, namely, the octavic correspondence of eighth and ninth base-points of pencils of cubics on seven given points, may be obtained from the Geiser transformation in space, the 7-ic correspondence of seventh and eighth base-points of quadrics on six given points; it arises from one of the invariant quadrics on the six points by simple projection of this quadric on a plane from one of the six points. Similarly, the 17-ic Bertini transformation may be obtained from T by projection of the correspondence set up by T on an invariant quadric Q from, say a_0.

§ 2. *The Quartic Surfaces C_i.*

A quartic surface with a triple point, a_0, in a given space is represented in a one-to-one way on a plane π in the same space by simple projection of its points from a_0. There are twelve lines through a_0 which lie on the surface; these are cut out by the cubic cone $\mathbf{\Gamma}_0$ tangent to the surface at a_0. On π the plane sections of C_0 appear as quartic curves on the points of intersection of a quartic and a cubic curve $\mathbf{\Gamma}$. The cubic $\mathbf{\Gamma}$ is the trace on π of the tangent cubic cone $\mathbf{\Gamma}_0$, and the quartic may be taken to be the intersection of C_0 and π. The quartic surface C_0 with triple point at a_0 is thus mapped from a plane π in its space by quartic curves on the intersections of a quartic and a cubic in π, and it is easily proved conversely that any such mapping scheme gives a quartic surface with a triple point. If two base-points of the system of quartic curves in π coincide at a point p, C_0 acquires a node, plane sections of C_0 through the node corresponding to the net of curves of the mapping system which have a node at p. In general this is the only way in which a node of C_0 can be represented. It follows from the mutual relations of π and the space figure that

f) *If a quartic surface have a triple point and a node, it has the same tangent plane at all points along the line joining the triple point and node.*

This result is so important for later developments that it seems advisable to add a direct geometrical proof. Let the surface have a triple point at a_0 and a double point at a_1 and consider the tangent plane at any point, x, of the line joining a_0 and a_1. The tangent plane meets the surface in the line a_0a_1 and a residual cubic curve which has a node at a_0 and passes through x and a_1; thus this residual cubic meets a_0a_1 in four points, and the line a_0a_1 must factor from it. The tangent plane at x therefore meets C_0 in the line a_0a_1 counted doubly, and hence must touch C_0 at all points along a_0a_1. Similarly, it may be proved that *if a surface of order n have two multiple points of multiplicities whose sum is $n+1$ it has the same tangent plane at all points along the line joining the multiple points.* The cubic surface with more than one node is a case in point. Likewise, *if a surface of order n have two points whose multiplicities total $n+2$ the line joining the points is a double line of the surface.* And so on.

In our mapping scheme the pencil of lines on π through p corresponds to plane sections of C_0 through triple point and node, and the tangent line to Γ at p corresponds to the conic cut from C_0 by the tangent plane along the line joining triple point and node.

For the surface C_0 of § 1 the section by a plane π is a quartic curve $C_{0\pi}$ touching at six points the section $\Gamma_{0\pi}$ of the cubic cone Γ_0 tangent to C_0 at a_0, the six points being the intersections of π with p_{0i} $(i=1, \ldots, 6)$. Hence,

g) *A plane π meets C_0 in a quartic curve which touches the intersection $\Gamma_{0\pi}$ of π with Γ_0 at the points $p_{0i}\pi$ $(i=1, \ldots, 6)$. The surface C_0 is then mapped from π by means of quartic curves touching $\Gamma_{0\pi}$ at the points $p_{0i}\pi$, and a quartic curve of this system in π is the projection of its corresponding plane section from a_0.*

A curve $r_i (i \neq 0)$ meets C_0 in all multiple points but the node a_i; these count for $5 \cdot 2+3=13$ intersections. Hence,

g′) *The surface C_0 contains the six cubic curves $r_i (i \neq 0)$. These are maps from π of conics on five out of the six contacts of the quartic curves with $\Gamma_{0\pi}$ on π.*

The latter statement is obvious from the space figure. Similarly,

g″) *The surface C_i contains the cubic curves $r_j (j \neq i)$.*

We indicate by $q_{0,1}$ the tangent line to r_1 at a_0 and by $q_{0,12}$ the tangent line to r_{12} at a_0.

Since C_1 contains the line p_{01} and the curves $r_2, \ldots, r_6$ it follows that the lines $p_{01}, q_{0,2}, \ldots, q_{0,6}$ are lines of a quadric cone with vertex at a_0, this cone being the cone tangent to C_1 at a_0. Again, projecting r_1 from a_0 we obtain a quadric cone containing the lines $q_{0,1}, p_{02}, \ldots, p_{06}$. For later reference we state the above in the theorem

h) *The lines p_{0i}, $q_{0,i}$ passing through a_0 are twelve generators of the cubic cone Γ_0 tangent to C_0 at a_0. The lines $q_{0,1}, p_{02}, \ldots, p_{06}$ are generators of a quadric cone, and the lines $p_{01}, q_{0,2}, \ldots, q_{0,6}$ are generators of a quadric cone. Analogous results follow by symmetry.*

§ 3. *The* (1, 4) *Quadratic Correspondence.*[*]

In what follows we shall be concerned mainly with the geometry of the bundle of lines through a_0; in order to attain clearer statement we shall regard this bundle as cut by a plane π. In this bundle it is important to distinguish between *directions* about a_0 and *lines* through a_0; these we shall denote by the generic symbols d and e, respectively. The symbols d and e will also be used to indicate the corresponding points in π, and the transformations P, R, R_{ijk}, R_{ij} to be defined are always to be taken as meaning the replacing of a d by one or more e's; their inverses replace e's by d's.

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[*] The reader will, of course, recognize in this the (1, 2) connex; cf. Clebsch-Lindemann, "Vorlesungen über Geometrie," pp. 1007-1014.

The most general (1, 4) quadratic correspondence in the plane is given by a set of equations of the form

$$\rho y_i = f_i(x), \tag{1}$$

where f_i are quadratic functions of x_1, x_2, x_3. From (1) we have for a given x a unique y, and this without exception if the conics $f_i=0$ have no common point; if $f_i=0$ pass through a point a, y is indeterminate for this point, and a is a singular point of the correspondence. Fixing y we have, eliminating ρ from (1),

$$\left\| \begin{matrix} y_1 & y_2 & y_3 \\ f_1 & f_2 & f_3 \end{matrix} \right\| = 0,$$

or if $K_1 = y_2 f_3 - y_3 f_2$, etc.,

$$K_1 = K_2 = K_3 = 0,$$

three conics of a pencil since

$$y_1 K_1 + y_2 K_2 + y_3 K_3 = 0;$$

the base-points of this pencil are the correspondents of y, four in number. The fixed points of (1) are given by the matrix

$$\left\| \begin{matrix} x_1 & x_2 & x_3 \\ f_1 & f_2 & f_3 \end{matrix} \right\| = 0,$$

of order 7. Hence,

i) *The general* (1, 4) *quadratic correspondence in the plane has seven fixed points.*

These fixed points can not all lie on a line; for this to occur the line would necessarily contain a singular point, and singular points do not exist in general. It follows by symmetry that no three fixed points can lie on a line. If three fixed points are chosen as the vertices of a triangle of reference, and the conics corresponding to the three sides of this triangle are $\phi_i=0$, the transformation appears in the form

$$\rho y_i = \phi_i(x) \tag{2}$$

where $\phi_i=0$ are now conics passing through the vertices of the triangle of reference by twos. The ϕ's are three linearly independent conics generating the net of conics corresponding to lines $(\eta y)=0$ of the plane.

Returning now to the figures in space, attention being fixed on the geometry about a_0, and on the section of the figures arising by a cutting plane, it is clear that there is a unique curve E, without exception, having a given direction d at a_0, this direction fixing on a plane π a point. The direction d, or the point in π, determines four lines e_i, the lines to the points where the corresponding curve E meets π. There is determined a $(1, 4)$ correspondence in π; to a given d correspond four e's. Call this P. In order to determine the order of P we consider a plane α through a_0. There is a unique quadric Q_a touching α at a_0, and all curves E with directions $a_0\alpha$ lie on Q_a; the contrary would mean too many intersections with Q_a. Hence, the transform by P of the line l which a marks on π is the intersection with π of the quadric Q_a, a conic.

In this and the sections immediately following we call the intersections of p_{0i}, a_0a, $q_{0,i}$, $q_{0,ij}$ with a cutting plane π, simply p_i, p, q_i, q_{ij}, respectively, where no ambiguity can arise. If the direction d at a_0 is p_{0i} the curve E determined is p_{0i} taken with the cubic curve on all a's but a_0 and a_i. The four points corresponding to p_i are the intersections of π with this curve and p_i itself. The points p_i and similarly p are therefore fixed points of P. We have

j) *The correspondence P determined on a plane π by making to correspond to a point x in π the four points of intersection of π with the curve E tangent to a_0x at a_0, is a quadratic* $(1, 4)$ *correspondence having the seven fixed points $p_1, \ldots, p_6, p$. The net of conics corresponding to lines $(\eta y) = 0$ of π is determined as the conics of intersection of π with quadrics Q.*

In the general $(1, 4)$ quadratic correspondence (1) let us suppose given seven pairs of corresponding points, say $x^{(i)}$, $y^{(i)}$, and let it be further required that the conics corresponding to three lines $\alpha^{(i)}$ pass through three definite points $a^{(i)}$, given in general position. This gives the set of twenty-four linear equations

$$\rho_j y_i = f_i(x^{(j)}), \quad j = 1, \ldots, 7, \quad i = 1, \ldots, 3$$
$$\sum_{i=1}^{3} \alpha_i^{(j)} f_i(x^{(j)}) = 0, \qquad j = 1, \ldots, 3.$$

This set of linear equations determines uniquely the ratios of the eighteen coefficients in f_i and the seven proportionality factors ρ_j. Hence,

k) *The most general* $(1, 4)$ *quadratic correspondence in the plane is uniquely determined when* (1) *seven pairs of corresponding points are given, and* (2) *the correspondents of three given lines are required to pass respectively through three points given in general position.* In particular,

k′) *The most general* $(1, 4)$ *quadratic correspondence in the plane is uniquely determined when its seven fixed points are given, three of these points,*

$p_1p_2p_3$ *are isolated, and it is required that the correspondent of* p_ip_j *pass through three points* a_k *in general position.*

The figure of eight base-points of a net of quadrics in space taken with a general plane π has $21+3-15=9$ absolute invariants. The complete figure determined in the plane π has therefore nine absolute invariants and $8+9=17$ constants. On the other hand the most general (1,4) quadratic correspondence in the plane has seventeen constants (the ratios of the eighteen coefficients in equations (1), and we would expect the correspondence P which we have determined to be general. It will now be proved that it is general, and a still more important result follows from the proof. First, the set of seven points p in π is general. For, choosing a plane π in space and seven general points p_ip on it, choose the point a_0; the lines $p_{01}, \ldots, p_{06}, p_0=a_0a$ are then determined. Choose a_4, a_5, a_6, a arbitrarily on $p_{04}, p_{05}, p_{06}, p_0$. There are then three unique quadrics Q_1, Q_2, Q_3 containing $p_{02}p_{03}a_4, \ldots, a$, etc., respectively. The quadrics Q_1, Q_2 and Q_3 meet p_{01}, p_{02}, p_{03} in three points a_1, a_2, a_3, respectively, and the points a thereby determined are the base-points of a net of quadrics of which Q_i are three linearly independent surfaces.* Hence,

l) *The set of points* p *in a plane* π *not passing through* a_0 *is general. The geometry in the bundle of lines at* a_0 *is identical*† *with the geometry of seven points in the plane.*

Let us now assume a general (1, 4) quadratic correspondence P' in a plane π in the form (2); The curves $\phi_i=0$ are conics passing through two out of three vertices of the reference triangle. Let these vertices be p_1, p_2, p_3. When the conics ϕ_i are given the equations (2) are not definite, but they are absolutely fixed when a fourth fixed point, say p_4, is given. The transformation P' is thereby uniquely determined. Choose now a point a_0 not on π and a point a_4 on a_0p_4. There is a unique quadric Q_1 containing the lines a_0p_1, a_0p_2, a_0p_3, the conic ϕ_1, and passing through a_4. Two other quadrics, Q_2, Q_3, are similarly determined. These quadrics are linearly independent. They pass through a_0, three points a_1, a_2, a_3 on a_0p_1, a_0p_2, a_0p_3 respectively, through a_4, and through three further points a_5, a_6, a. The (1, 4) correspondence determined on π by the points a with a_0 isolated must now have exactly the equations (2), and must coincide with P'. Hence,

m) *The geometry of the* (1, 4) *quadratic correspondence in the plane is completely identical with that of the plane section of the figure of eight base-*

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* Compare A. C. Dixon, *Quarterly Journal*, Vol. XLI (1910), p. 210 ff.

† The word "identical" is used here in a slightly strained sense, but its meaning will not be misunderstood.

points of a net of quadrics when one of these base-points is isolated. The domains of rationality are in both cases the same. Or,

m′) *The geometry of the* (1, 4) *quadratic correspondence is identical with that of the configuration which is the plane section of a complete seven-point in space. The fixed points of the correspondence pair with the seven points.*

It follows easily from the argument above that *the* (1, 4) *quadratic correspondence is uniquely determined when the net of conics corresponding to lines of the plane and four fixed points are given.* For, if $p_1, \ldots, p_4$ are the four fixed points, $p_1, \ldots, p_3$ may be chosen as vertices of the reference triangle; then there is a unique conic of the net passing through $p_2\, p_3$; this is the ϕ_1 of (2). The conics ϕ_2 and ϕ_3 are similarly determined. The fourth fixed point p_4 fixes the arbitrary multipliers of the ϕ's in (2), and the correspondence is uniquely determined. It is obviously immaterial which three p's are chosen as vertices of the triangle of reference. This may be stated in another way.

m″) *The* (1, 4) *quadratic correspondence is uniquely determined by a cubic curve and four points in general position given as fixed points of the correspondence;* the polars of the cubic, of course, furnishing the net of conics.

It is of interest to see how the configurations of theorems m) and m′) determine the correspondence without reference to space. For brevity, we omit proofs of the statements made; little further use is to be made of them. The configuration of theorem m′) consists of thirty-five lines and twenty-one points, in which every line carries three points and every point five lines. The points and lines may be named by symbols ij and ijk, respectively, chosen from seven indices, the point ij being incident with the line ijk. Now, if five indices are chosen from the seven and we consider only those points and lines whose names involve indices among these, we obtain a Desargues configuration (configuration B). There are twenty-one such configurations contained in the larger configuration. A configuration B is self-polar with regard to a definite conic N; we obtain in this way twenty-one conics N_{ij}, and these conics are in a linear system, they lie in a web of conics. To this web is apolar a net of conics, and this net is the net of our correspondence. The seven fixed points may be linearly constructed from our configuration, and the correspondence P is thereby determined.

It is easy to see from the space figure how, conversely, the configuration is determined when P is given. We show in the next section how the points q_i and q_{ij} are determined by the fixed points of P alone. Let us consider a point 12, say, of the configuration, plane and space figures being supposed given. The line $a_1 a_2$ marks on π the point 12. This line taken with r_{12} is a degenerate curve E, and the tangent to r_{12} at a_0 marks on π the point q_{12}. Thus *the transform by* P *of* q_{12} *is the point* 12 *of the configuration taken with the three points in which* r_{12} *meets* π. The points of the configuration thus appear as transforms by P of the points q_i, q_{ij}, whose importance will be abundantly evident in what follows. This may be stated purely from the point of view of the plane correspondence. Given P the seven fixed points are given and the points q_i, q_{ij} are determined by them. To q_{12}, q_{23}, q_{31} correspond by P three sets of four points. Three points may be selected, one from each set of four, and in only one way, such that the three points lie on a line. These points are the points 12, 23, 31 of the configuration.

In the figure in space, as we have remarked, there is a one-one correspondence between quadrics Q and planes on a_0, every quadric having a unique tangent plane at a_0 and every plane on a_0 having a unique tangent quadric. On the quadric is a pair of lines which lie in the tangent plane. Projecting from a_0, the curves E on Q give a pencil of cubics in π on the points p and x, y, —the traces on π of the pair of lines in the tangent plane. In the plane π the points x, y are pairs of the Geiser involution, G, of order 8, having the p's as triple singular points, the correspondent of p_1 say, being the cubic with node at p_1 and on $p_2, \ldots, p$. There is a unique pair of points x, y—pairs of G—on any line; this, of course, may be seen from the space figure.* But since a quadric Q marks on π the transform by P of the trace of its tangent plane at a_0 (Th. j)) and the points x, y are on this line, we have

n) *The pair of points in which a line meets its correspondent conic in a* (1, 4) *quadratic correspondence,* P, *are partners in the Geiser involution determined by the seven fixed points of* P.

This is true without exception; on a line through p_1 say, p_1 and the point where the line meets the corresponding cubic are partners. This is still true, if the line passes through another singular point, p_2 say. On p_1p_2, p_1 and p_2 are to be regarded as partners.

§ 4. *The Transformations* R_{ijk} *and* R_{ij}.

We saw in the previous section that a (1, 4) quadratic correspondence is determined in the bundle at a_0 or on a plane π, and if π is arbitrary (or in general position), the (1, 4) correspondence is general. This may be stated in another way. A cutting plane π may be fixed once for all, and all possible (1, 4) correspondences with the seven general fixed points may be studied by regarding them as determined by a second plane π' in any position, and by choosing π' in a special position we obtain a specialization of P. After all we are studying merely the geometry at a_0, the planes of space figuring as auxiliaries. In this section we examine the most interesting of these specializations.*

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* Compare Dixon, *loc. cit.*

The general quadratic Cremona transformation in the plane is uniquely determined when three singular points (the three of either set) and its four fixed points are given, for, if the singular points are taken as reference points, the equations of the transformations are of the form

$$\rho y_i = \sum_j \alpha_j^{(i)}/x_j,$$

and when four fixed points are given we have from this twelve equations to determine the ratios of the nine α's and the four proportionality factors. Thus, it is clear, *a priori*, that given seven general points in the plane, a set of thirty-five quadratic Cremona transformations is determined by requiring three of the seven points to be singular and the other four fixed. We consider the set of thirty-five transformations determined in this way by the seven points p in the plane π. For the immediate use which we make of these transformations, it is convenient to separate them into a set of twenty, and a set of fifteen. We call the transformation with singular points p_1, p_2, p_3 and fixed points p_4, p_5, p_6, p, R_{123}^{-1}, and the transformation with singular points p_1, p_2, p and fixed points p_3, p_4, p_5, p_6, R_{12}^{-1}. There are twenty transformations of the first type and fifteen transformations of the second. The inverse notation is here chosen in order to conform to the notation of the previous section. These transformations are specializations of the (1, 4) correspondence P.

Consider the section of the space figure by the plane π_{123} on a_1, a_2 and a_3. Lines in the bundle on a_0 give us points on π_{123}, thus we have from $p_{0i}a_0a$ a set of seven points p_1, p_2, p_3 $(=a_1, a_2, a_3)$, p_4, p_5, p_6, p; from $q_{0,i}$ a set of six points q_i, and from $q_{0,ij}$ a set of fifteen points q_{ij}. Given a direction d at a_0 a curve E is determined which meets π_{123} in p_1, p_2, p_3 (a_1, a_2, a_3) and a fourth point e. The correspondence (d, e) is here one-to-one; call it R_{123}. It appears here geometrically as a special case of the correspondence P of the previous section, and is therefore a quadratic Cremona transformation. The transform $R_{123}\, l$, where l is a line in π_{123}, is a conic on p_1, p_2, p_3, and hence, p_1, p_2, p_3 are singular points of R_{123}^{-1}; this transformation is identical with the R_{123} of the previous paragraph. The singular points of R_{123} are the correspondents by

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* Compare Clebsch-Lindemann, *loc. cit.* See also Timerding, *Mathematische Annalen,* Vol. LIII, p. 193 ff.

R_{123}^{-1} of the lines $p_2 p_3$, $p_3 p_1$, $p_1 p_2$. Now, $p_2 p_3$ taken with r_{23} is a degenerate curve E; hence, the correspondent of $p_2 p_3$ by R_{123}^{-1} is marked on π_{123} by the tangent $q_{0,23}$ to r_{23} at a_0; that is, it is the point q_{23}. This gives us

o) *The transformation* R_{123} *is a quadratic Cremona transformation with singular points* q_{23}, q_{31}, q_{12}; R_{123}^{-1} *has singular points* p_1, p_2, p_3. *The transform* $R_{123}\, l$ *where* l *is any line meets* l *in a pair of the Geiser involution* G *determined by the seven points* $p_1, \ldots, p_6, p$. *It has the four fixed points* p_4, p_5, p_6, p.

The two latter statements of this theorem are consequences of the fact that R_{123} is a specialization of P. For the first see Theorem j), for the second, Theorem n). Of course, they may be proved directly by the methods employed in the proof of the theorems cited. There are twenty transformations R_{ijk} determined in a manner similar to the above; R_{ijk} has singular points q_{jk}, q_{ki}, q_{ij}, with corresponding fundamental lines $p_j p_k$, $p_k p_i$, $p_i p_j$: R_{ijk}^{-1} has singular points p_1, p_2, p_3 with corresponding fundamental lines q_{12}, q_{13}, etc. The transformation has all p's, except p_i, p_j, p_k as fixed points.

We consider now the three points $R_{123}q_i$ $(i=1, 2, 3)$. The point q_i is the trace on π_{123} of the tangent to r_i at a_0. Hence $R_{123}q_i$ is the third point, *i. e.*, the point not at p_j or p_k, in which r_i meets π_{123}. We call this point q_i^{123}; there are three such points. Since r_i lies on the quartic surface C_0 (see p. 162), it follows that the points q_i^{123} lie on the section of C_0 by π_{123}; this is a rational quartic with nodes at p_1, p_2 and p_3, C_{123} say. Now C_0 contains the lines $a_0 a_i$, hence, C_{123} passes through the points p_4, p_5, p_6—fixed points of R_{123}. Transforming C_{123} by R_{123}^{-1} and remembering that q_i^{123} passes into q_i we obtain the theorem:

p) *The six points* q_1, q_2, q_3, p_4, p_5, p_6 *are six points of a conic, the transform* $R_{123}^{-1}C_{123}$. *Similarly the six points* q_i, q_j, q_k, p_l, p_m, p_n *are six points of a conic, the transform* $R_{ijk}^{-1}C_{lmn}$.

The twelve points $p_1, \ldots, p_6, q_1, \ldots, q_6$ are seen from Theorems h) and p) to lie by sixes on thirty-two conics, Theorem h) furnishing twelve, and Theorem p) furnishing twenty. Observe that there are sixteen pairs of conics, any pair containing the whole set of twelve points. These facts will appear later in another light.

After the preceding discussion of R_{123} the transformations which we call R_{ij} are easily disposed of. We consider the plane π_{12} on a_1, a_2, a. As before, we indicate the projections of a_i, a on π_{12} by p_i, p. Directions d determine on π_{12} sets of four points e of which three are at p_1, p_2, p; the fourth is variable. Again we have a quadratic Cremona transformation, a special case of P, which we call R_{12}. By an argument similar to that employed in the case of R_{123} we have

q) *The quadratic Cremona transformation* R_{12} *has the singular points* q_1, q_2, q_{12} *and the fixed points* p_3, p_4, p_5, p_6. R_{12}^{-1} *has singular points* p_1, p_2, p. *The points* q_1, q_2, q_{12} *pair respectively with* p_1, p_2, p.

§ 5. *The Quintic Cremona Transformation R.*

It is for its usefulness in connection with the transformation which is the subject of this section that the division of the quadratic transformations of the previous section into two sets is adopted for the present; of course all thirty-five are a symmetrical set determined by seven points.

The 15-ic Cremona transformation T of § 1 has the singular point a_0, and to a_0 corresponds the quartic surface C_0; this surface has a triple point at a_0 and double points at $a_1, \ldots, a_6$ and, as we saw, is uniquely determined by these conditions. To directions about a_0 correspond univocally points of C_0; to a direction d at a_0 corresponds a definite line e on a_0, the line joining a_0 to the correspondent d on C_0. Conversely, given the line e on a_0, it meets C_0 at only one point not at a_0 since a_0 is a triple point of C_0, and this point furnishes a unique correspondent d. We obtain thus a Cremona transformation (d, e) in a cutting plane π which we denote by R.

Quadrics Q are invariant under T. The directions along a quadric Q at a_0 must therefore pass by T into the curve of intersection of the quadric with C_0. Let l be a line in a plane π and let Q_l be the unique quadric Q touching the plane $a_0 l = \alpha$ at a_0. Since α touches Q_l at a_0 and therefore defines there the same set of directions, it follows that to the directions $a_0\alpha$ correspond points on the curve of intersection of Q_l and C_0, a curve of order 8 with triple point at a_0 and double points at $a_1, \ldots, a_6$. Projecting this octavic curve from a_0, its triple point, we obtain in π the curve Rl, a quintic curve with nodes at $p_1, \ldots, p_6$; R is, therefore, a quintic Cremona transformation and $p_1, \ldots, p_6$ are the singular points of R^{-1}. Since Q_l touches α at a_0 the curve of intersection of Q_l and C_0 touches α along three directions at a_0, and these directions lie along the generators of Γ_0, the cubic cone tangent to C_0 at a_0. Since J, the surface of invariant points of T, has at a_0 the tangent cone Γ_0 (Th. d)) d and e must coincide along these directions, and the transformation R in π has Γ, the trace of Γ_0 on π, as a curve of fixed points.

The curves r_i lie entirely on C_0 and to a point of such a curve corresponds by T the whole curve. To the direction d at a_0 along $q_{0,i}$, the tangent to r_i at a_0, corresponds the whole curve r_i, and hence to the point q_i in π corresponds by R the projection of r_i from a_0, the conic on the five points $p_j (j \neq i)$. The points $q_1, p_2, \ldots, p_6$ are on a conic (Th. h)), u_1 say. Five other conics are similarly defined. The points q_i thus fixed on Γ when the p's are given are singular points of R. The corresponding fundamental curves are of course the conics u_i, but it is also especially to be noted that the conic $q_1, \ldots, q_5 = v_6$, say, passes through p_6 (Th. h)). It thus appears that the points p_i and q_i are two symmetrical sets, although our determination of them is asymmetrical. That the two sets are symmetrical will be proved presently. Summarizing the above we have the theorem:

r) *The transformation R determined on a plane π by the space figure is quintic. R has the singular points $q_1, \ldots, q_6$, and R^{-1} the singular points $p_1, \ldots, p_6$. The twelve points $p_1, \ldots, p_6, q_1, \ldots, q_6$ lie on a cubic curve Γ which is a curve of fixed points of R, and R has besides the fixed point p not on Γ* (Th. e)). *It has no other fixed points. The transformation R is the most general quintic Cremona transformation in the plane to within a collineation on the q's. The conic u_1 on $p_2, \ldots, p_6$ passes through q_1, and the conic v_1 on $q_2, \ldots, q_6$ passes through p_1. There are two sets of six such conics which are fundamental curves corresponding to the singular points of R and R^{-1}.*

Two statements in this theorem require proof. That R is general to within a collineation on the q's follows from the fact that the set of points p_i is general. The statement that R has no fixed points other than those cited may be proved as follows: The general (1, 25) quintic correspondence in the plane may be given in the form

$$\rho y_i = f_i(x), \tag{1}$$

where f_i are general homogeneous functions of the coordinates x_i of order 5. The fixed points of this correspondence are the roots of the matrix

$$\left\| \begin{matrix} x_1, & x_2, & x_3 \\ f_1, & f_2, & f_3 \end{matrix} \right\| = 0, \tag{2}$$

thirty-one in number. If (1) is a Cremona transformation the three quintics f_i have six common nodes, and the three sextics of the matrix (2) have the same nodes. The number of fixed points in the case of the general quintic Cremona transformation is, therefore, $31 - 6 \cdot 4 = 7$. If (1) in addition has a cubic curve of fixed points Γ—passing necessarily through both sets of singular points—the curve Γ is a factor of the three determinants (2). The two cubics

$$\frac{x_1 f_2 - x_2 f_1}{\Gamma} = 0, \quad \frac{x_1 f_3 - x_3 f_1}{\Gamma} = 0$$

meet in the six singular points, the two points $x_1 = f_1 = 0$ (the three points of

$x_1=f_1=0$ on Γ are to be excluded) and in one further point. There is, therefore, only one fixed point of the transformation which does not lie on Γ. It is here especially to be emphasized that

s) *The transformation R is the most general quintic Cremona transformation in the plane that has a cubic curve of fixed points.*

If μ_i are the elliptic parameters of p_i on the curve Γ, these parameters must be taken to be connected by the relation (Th. g))

$$\Sigma\mu\equiv\omega,$$

where ω is a half period of the elliptic functions on Γ. If ν_i are the parameters of q_i, we have, since q_i and the five points p_j $(j\neq i)$ are on a conic,

$$\omega-\mu_i+\nu_i\equiv0,$$

whence

$$\nu_i\equiv\mu_i-\omega,$$

an involutory relation connecting μ_i and ν_i. Also

$$\Sigma\nu_i\equiv\Sigma\mu_i\equiv\omega,$$

and the symmetry of the relations between the p's and q's is put definitely in evidence, either set given on Γ determining the other uniquely and in the same way.

The line p_1p_2 meets Γ again in the point with parameter $-(\mu_1+\mu_2)$ and the line q_1q_2 passes through the same point. Therefore,

t) *The lines p_ip_j and q_iq_j meet on the cubic curve Γ.*

It may easily be verified that the lines p_1q_2 and p_2q_1 meet on the cubic Γ, and that there are thus obtained fifteen points like 12, lying by threes on fifteen lines like $12|34|56$, this being the line through the three points 12, 34, and 56. The configuration thus obtained is the section by π of the fifteen lines and fifteen tritangent planes of a cubic surface left when a double six is omitted. The transforms of these lines by R bear especially interesting relations to the lines themselves.*

§ 6. *Analytical Form of the Transform of a Curve by R.*

It is possible, and in a unique way, to build up from the quadrics Q a surface of order $2n$ having the points a as n-fold points when the tangent cone at one of these points is given arbitrarily. Let the a chosen be a_0 and choose a_0 as the vertex $x_1=x_2=x_3=0$ of the tetrahedron of reference, the opposite plane being chosen as the plane π of former sections. Let $(\alpha x)^n=0$ be a given cone at a_0, —this cone is of course uniquely determined by the curve $(\alpha x)^n=0$ in π, $(\alpha x)^n=0$ not involving x_4. Let Q_1, Q_2, Q_3 be the three quadrics Q

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* See Morley, *Johns Hopkins Circulars*, February, 1912, p. 69.

touching the three reference planes at a_0. These quadrics are linearly independent. Then

$$(\alpha Q)^n = 0 \tag{1}$$

is the surface above cited having the tangent cone $(\alpha x)^n = 0$ at a_0. For, the Q's are of the form

$$\left.\begin{aligned} Q_1 &\equiv x_1 x_4 - k_1, \\ Q_2 &\equiv x_2 x_4 - k_2, \\ Q_3 &\equiv x_3 x_4 - k_3, \end{aligned}\right\} \tag{2}$$

where k_i are quadratic functions of x_1, x_2, x_3. Substituting (2) in (1), we find that the coefficient of x_4^n, the highest power of x_4 occurring, is $(\alpha x)^n$, and hence, (1) is the surface built on the Q's of order $2n$ and with n-fold point at a_0 and the tangent cone $(\alpha x)^n = 0$.

The k's in (2) are significant as being the intersections of the quadrics Q_i with $\pi (x_4 = 0)$. Since Q_i touches the plane x_i at a_0 $(i = 1, 2, 3)$ we have

$$P x_i = k_i, \tag{3}$$

where P is the quadratic (1, 4) correspondence of § 3.

The surface $(\alpha Q)^n = 0$ may be defined geometrically in such a way as to bring out more clearly the matters with which we are dealing. Let, in the first place, a curve $(\alpha x)^n = 0$ be given in π, and let it be regarded as defining a set of directions at a_0. The curves E with these directions lie on a surface whose intersection with π is $P(\alpha x)^n$, a curve of order $2n$. This surface must coincide with $(\alpha Q)^n = 0$. Incidentally,

u) *Every locus of curves E is of the form* $(\alpha Q)^n = 0$, *and, conversely, every surface of this form is a locus of curves E.*

This might also have been inferred from the facts that, 1) a curve E meets $(\alpha Q)^n = 0$ in $8n$ points; 2) that these $8n$ intersections are accounted for at the points a; 3) that the sum of the elliptic parameters on any curve E at the points a is a period; 4) that, therefore, if a curve E meets $(\alpha Q)^n = 0$ at a point not an a, it lies entirely on the surface.

The invariance under T of the surface $(\alpha Q)^n = 0$ follows from the fact that it is a locus of curves E, and of course this is also obvious from its form. In order to determine the transform by R of $(\alpha x)^n = 0$ we have only to project from a_0 the curve of intersection of $(\alpha Q)^n = 0$ with C_0, where the Q's are of the form (2). We have from (2)

$$(\alpha Q)^n \equiv [(\alpha x) x_4 - (\alpha k)]^n = 0, \tag{4}$$

or, taking account of (3), and writing

$$\alpha_1 P x_1 + \alpha_2 P x_2 + \alpha_3 P x_3 = (\alpha P x) = P(\alpha x), \quad (\alpha Q)^n \equiv [(\alpha x) x_4 - P(\alpha x)]^n. \tag{5}$$

Now, C_0 is of the form

$$C_0 \equiv \mathbf{\Gamma} x_4 + C = 0, \tag{6}$$

where $C=0$ is the quartic curve of intersection of C_0 with π ($x_4=0$); then the projection of the curve of intersection of C_0 and $(\alpha Q)^n=0$, or the transform $R(\alpha x)^n$ is obtained by eliminating x_4 from (5) and (6). Substituting in (5) the value of x_4 from (6), we obtain

$$R(\alpha x)^n \equiv (\alpha x)^n C^n + \binom{n}{1} (\alpha x)^{n-1} P(\alpha x) C^{n-1} \mathbf{\Gamma} + \binom{n}{2} (\alpha x)^{n-2} P(\alpha x)^2 C^{n-2} \mathbf{\Gamma}^2 \\ + \dots\dots + P(\alpha x)^n \mathbf{\Gamma}^2 = 0.^* \tag{7}$$

We state this in the theorem

v) *Given the transformations* R, R^{-1} *having the singular points* q_i, p_i, *the curve of fixed points* $\mathbf{\Gamma}$ *and the fixed point* p, *and choosing any one of the* ∞^3 $(1, 4)$ *quadratic correspondences, say* P, *with the fixed points* $p_i p$, *the transform* $R(\alpha x)^n$, *where* $(\alpha x)^n=0$ *is any curve in the plane, is expressible in terms of the cubic* $\mathbf{\Gamma}$, *a quartic* C *determined uniquely by* P, *and the set of covariants of* $(\alpha x)^n$ *and* P *which are of the form*

$$(\alpha x)^{n-R} P(\alpha x)^R = 0, \quad (R = 0, 1, \dots, n). \tag{8}$$

These curves are loci of points x *which lie on the correspondent by* P *of their* $(n-R)$*-th polars as to* $(\alpha x)^n=0$.

The last statement is easily proved. Fixing x, the $(n-R)$-th polar of x to $(\alpha x)^n=0$ is $(\alpha x)^{n-R}(\alpha y)^R=0$, where y is variable; its transform by P is $(\alpha x)^{n-R} P(\alpha y)^R = 0$, and the condition that x should lie on this curve is $(\alpha x)^{n-R} P(\alpha x)^R = 0$.

Applying (7) to the cubic $\mathbf{\Gamma} \equiv (\gamma x)^3$ which we know to be invariant under R, and remembering that if a curve passes through a singular point q_i of R, the conic u_i factors from its transform, we obtain

$$R\mathbf{\Gamma} \equiv u_1 u_2 \dots . u_6 \mathbf{\Gamma} \equiv \mathbf{\Gamma} C^3 + 3(\gamma x)^2 P(\gamma x) \mathbf{\Gamma} C^2 + 3(\gamma x) P(\gamma x)^2 \mathbf{\Gamma} C^2 + P\mathbf{\Gamma} \cdot \mathbf{\Gamma}^2,$$

whence the interesting identity

$$u_1 \dots . u_6 \equiv C^3 + 3(\gamma x)^2 P(\gamma x) C^2 + 3(\gamma x) P(\gamma x)^2 C\mathbf{\Gamma} + P\mathbf{\Gamma} \cdot \mathbf{\Gamma}^2. \tag{8$'$}$$

The curve $P\mathbf{\Gamma}$ is a sextic passing through the points p_i, since p_i are fixed points of P. The curve u_i meets $C=0$, a quartic touching $\mathbf{\Gamma}$ at the points p_i, at three points not on $\mathbf{\Gamma}$, $u_i=0$ passing through the five points p_j $(j \neq i)$ of $\mathbf{\Gamma}$. The identity (8′) tells us that the eighteen points thus obtained from u_i are the $24-6=18$ intersections of $P\mathbf{\Gamma}$ and C which are not on $\mathbf{\Gamma}$. This may be seen in another way, which incidentally serves to identify these six triads of points. The transform of q_i by P is obtained from the direction d which $q_{0,i}$ determines at a_0, the corresponding curve E being r_i taken with the line $a_i a$. This meets

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* No attempt is here made to fix the possible constant multipliers of the forms occurring in this formula. This would become necessary if several choices of P were made and exact results were required from combination of the resulting expressions. It is sufficient for our present purpose that the formula (7) gives a system of curves from which $R(\alpha x)^n$ may be linearly built.

π in a point of the configuration of theorem m′) and in the three points in which r_i meets π. Since r_i lies on C_0 and is projected from a_0 into u_i, the inferences from the identity are obvious. The curve C is of course determined by these conditions.

As a special case of theorem v) we have

v′) *The transform of a line l of π by R is of the form*

$$Rl \equiv Cl + \Gamma Pl, \tag{9}$$

where P is any one of the $\infty^3\,(1,4)$ correspondences having p_i, p as fixed points.

We infer from (9) that, 1) Rl meets Γ at points of C, — C touches Γ at p_i and Rl has nodes there; 2) Rl meets l at its three points of intersection with Γ and its two points of intersection with Pl, —the former is obvious from the fact that points of Γ are fixed under R, the latter furnishes additional information; 3) Rl meets C at its eight intersections with Pl. We have here the new facts:

w) *The transform Rl meets l in five points, three of which are on Γ and are fixed points of R; the other two are the intersections of l with its transform by P. These two points are the pair of the Geiser involution determined by the points p_i, p which lie on l* (Th. n)). *It meets Pl in these two points and in its intersections with C, the quartic curve associated with P.*

Since the quadratic transformation R_{ijk} and R_{ij} are special cases of the correspondence P, we have

x) *The transform Rl may be written in twenty ways in the form*

$$C_{ijk}\, l + \Gamma R_{ijk}\, l = 0,$$

and in fifteen ways in the form

$$C_{ij}\, l + \Gamma R_{ij}\, l = 0.$$

Theorems v′) and x) lead to an endless variety of algebraic identities, one of which we use in the next section.

§ 7. *The Points q_{ij} and the Quadratic Transformations S.*

The transformation R sends points on the line q_iq_j into points on the line p_ip_j, and a correspondence is thus determined between the points of the two lines which is necessarily projective since it is univocal. The lines p_ip_j and q_iq_j meet in a point of Γ (Th. t)), a fixed point of R, and hence their intersection is self-corresponding. The ranges determined on p_ip_j and q_iq_j by R are, therefore, in perspective position. It will presently be seen that the point q_{ij} is the center of perspective, this theorem showing exactly how the points q_{0_j} may be determined directly by R without using the quadratic transformations R_{ijk}.

The transform Rl may be written in the form (Th. x))

$$Rl \equiv C_{123}l + \Gamma R_{123}l,$$

where C_{123} is the rational quartic cut from C_0 by the plane on a_1, a_2, a_3 (or its projection); C_{123} touches Γ at p_4, p_5, p_6 and has nodes at p_1, p_2, p_3, and is thereby uniquely determined: l is any line of the plane and $R_{123}l$ its transform by the quadratic transformation of § 4. R_{123} has singular points q_{23}, q_{31}, q_{12} and R_{123}^{-1} singular points p_1, p_2, p_3. Take l now as the line $q_{12}q_{13}$. We have

$$R\overline{q_{12}q_{13}} \equiv C_{123}\cdot\overline{q_{12}q_{13}}+\Gamma\cdot\overline{p_1p_2}\cdot\overline{p_1p_3}, \tag{1}$$

since $\overline{q_{12}q_{13}}$ contains the two singular points q_{12}, q_{13}, and hence

$$R_{123}\overline{q_{12}q_{13}} \equiv \overline{p_1p_2}\cdot\overline{p_1p_3}. \tag{2}$$

We see from the identity (1) that $R\overline{q_{12}q_{13}}$ meets $\overline{p_1p_2}$ at its intersection with $\overline{p_{12}p_{13}}$, and this point must be the transform by R of the point in which $\overline{q_{12}q_{13}}$ meets $\overline{q_1q_2}$. We have then that the line $\overline{q_{12}q_{13}}$ through q_{12} marks on $\overline{p_1p_2}$ and $\overline{q_2q_1}$ correspondents by R. Similarly, it may be proved that $\overline{q_{12}q_{14}}$, $\overline{q_{12}q_{15}}$, $\overline{q_{12}q_{16}}$ mark on these lines correspondents by R, and hence q_{12} is the center of perspective of these ranges. We state this in the theorem

y) *The fifteen points* q_{ij} *are symmetrically related to the two sets of seven points* $p_1, \ldots, p_6, p, q_1, \ldots, q_6, p$, *either set determining them in the same way. The point* q_{ij} *is the center of perspective of the correspondence determined between the lines* $\overline{p_ip_j}$ *and* $\overline{q_iq_j}$ *by the quintic Cremona transformation* R *which is itself determined by either set of seven points.*

From the symmetry of the relations of the points q_{ij} to the two sets of seven points p_i, p, q_i, p, we have

z) *There exist twenty quadratic Cremona transformations* S_{123}, *etc.*; S_{123} *has singular points* q_{23}, q_{31}, q_{12}; S_{123}^{-1} *has singular points* q_1, q_2, q_3, *and* S_{123} *has fixed points* q_4, q_5, q_6, p. *There exist fifteen quadratic Cremona transformations* S_{12}, *etc.;* S_{12} *has the singular points* q_{12}, p_1, p_2, *and* S_{12}^{-1}, *the singular points* p_1, q_1, q_2. *The transformation* S_{12} *has the fixed points* q_3, q_4, q_5, q_6.

In the part of this paper which is to follow the correspondences here treated will be discussed in connection with their relation to the nodes of a quartic of lines. The lines of the quartic referred to are marked on π by planes tangent at a_0 to nodal quadrics Q. This curve has the twenty-eight points p_i, p, q_i, q_{ij} as nodes. This known fact we shall prove by means of the transformations whose theory we have developed. The existence of a vast number of other transformations follows by symmetry, and these transformations have what seems to the author an interesting bearing on certain configurations of points and lines associated with the nodes, apart from the fact that they seem to add geometrical interest to their abstract grouping.

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